EJERCICIO 1 (33:53)

Para la teoría de Klein-Gordon, el operador campo en la teoría de Schrödinger es:

$$\boldsymbol{\hat{\phi}_{(\vec{x})} = \int \frac{dk}{2\pi}\frac{1}{\sqrt{2\omega_k}}\left(a_k e^{i\,\vec{k}\cdot\vec{x}} + a_k^\dagger e^{-i\,\vec{k}\cdot\vec{x}}\right)}$$

Calcular el operador campo, dependiendo del tiempo

$$\boldsymbol{\hat{\phi}_{(t\,,\vec{x})} = e^{i\,t\,H}\hat{\phi}_{(\vec{x})}e^{-i\,t\,H}}$$

Considerar que

$$e^{xA}\,B\,e^{-xA} = B + [A\,,B]\,x + \frac{1}{2!}[A\,,[A\,,B]]\,x^2 + \frac{1}{3!}\big[A\,,[A\,,[A\,,B]]\big]\,x^3 + \cdots$$

Vamos a calcular

$$e^{it\,H}\,a_k\,e^{-it\,H}$$

De modo que *it = x*; $a_k = B$ y *H=A* es el hamiltoniano de Klein-Gordon

$$H = \int \frac{d^3k}{(2\pi)^3}\,\omega_k a^\dagger_{(\vec{k})}\,a_{(\vec{k})}$$

Con los siguientes conmutadores:

$$\left[H\,,a_{(\vec{k})}\right] = -\omega_k\,a_{(\vec{k})}$$

$$\left[H\,,a^\dagger_{(\vec{k})}\right] = \omega_k\,a^\dagger_{(\vec{k})}$$

$$e^{it\,H}\,a_k\,e^{-it\,H} = a_k + [H\,,a_k]\,x + \frac{1}{2!}[H\,,[H\,,a_k]]\,x^2 + \frac{1}{3!}\big[H\,,[H\,,[H\,,a_k]]\big]\,x^3 + \cdots$$

$$[H\,,[H\,,a_k]] = [H\,,(-\omega_k\,a_k)] = -\omega_k\,[H\,,a_k] = -\omega_k(-\omega_k\,a_k) = \omega_k^2\,a_k$$

$$\big[H\,,[H\,,[H\,,a_k]]\big] = \left[H\,,(\omega_k^2\,a_k)\right] = \omega_k^2\,[H\,,a_k] = \omega_k^2\,(-\omega_k\,a_k) = -\omega_k^3\,a_k$$

$$e^{it\,H}\,a_k\,e^{-it\,H} = a_k - \omega_k\,a_k\,(it) + \frac{1}{2!}\omega_k^2\,a_k\,(it)^2 - \frac{1}{3!}\omega_k^3\,a_k\,(it)^3 + \cdots$$

$$e^{it\,H}\,a_k\,e^{-it\,H} = a_k + (-1)\omega_k\,a_k\,(it) + (-1)^2\frac{1}{2!}\omega_k^2\,a_k\,(it)^2 + (-1)^3\frac{1}{3!}\omega_k^3\,a_k\,(it)^3 + \cdots$$

$$e^{it\,H}\,a_k\,e^{-it\,H} = a_k\left(1 + \omega_k\,(-it) + \frac{1}{2!}\omega_k^2\,(-it)^2 + \frac{1}{3!}\omega_k^3\,(-it)^3 + \cdots\right)$$

$$e^{it\,H}\,a_k\,e^{-it\,H} = a_k\,e^{-i\,\omega_k\,t}$$

$$e^{it\,H}\,a_k^\dagger\,e^{-it\,H} = a_k^\dagger + [H\,,a_k^\dagger]\,x + \frac{1}{2!}\left[H\,,[H\,,a_k^\dagger]\right]\,x^2 + \frac{1}{3!}\left[H\,,\left[H\,,[H\,,a_k^\dagger]\right]\right]\,x^3 + \cdots$$

$$\left[H\,,\left[H\,,a_k^\dagger\right]\right]=\left[H\,,\left(\omega_k\,a_k^\dagger\right)\right]=\omega_k\left[H\,,a_k^\dagger\,\right]=\omega_k\left(\omega_k\,a_k^\dagger\right)=\omega_k^2\;a_k^\dagger$$

$$\left[H\,,\left[H\,,\left[H\,,a_k^\dagger\right]\right]\right]=\left[H\,,\left(\omega_k^2\,a_k^\dagger\right)\right]=\omega_k^2\left[H\,,a_k^\dagger\right]=\omega_k^2\left(\omega_k\,a_k^\dagger\right)=\omega_k^3\,a_k^\dagger$$

$$e^{it\,H}\,a_k^\dagger\,e^{-it\,H}=a_k+\omega_k\,a_k^\dagger\,(it)+\frac{1}{2!}\omega_k^2\;a_k^\dagger\,(it)^2+\frac{1}{3!}\omega_k^3\,a_k^\dagger\,(it)^3+\cdots$$

$$e^{it\,H}\,a_k^\dagger\,e^{-it\,H}=a_k^\dagger\left(1+\omega_k\,(it)+\frac{1}{2!}\omega_k^2\,(it)^2+\frac{1}{3!}\omega_k^3\,(it)^3+\cdots\right)$$

$$e^{it\,H}\,a_k^\dagger\,e^{-it\,H}=a_k^\dagger\,e^{i\,\omega_k\,t}$$

También podría hacerse

$$\left(e^{it\,H}\,a_k\,e^{-it\,H}=a_k\,e^{-i\,\omega_k\,t}\right)^\dagger$$

$$\left(e^{it\,H}\,a_k\,e^{-it\,H}\right)^\dagger=\left(e^{-it\,H}\right)^*a_k^\dagger\left(e^{-it\,H}\right)^*=e^{it\,H}\,a_k^\dagger\,e^{-it\,H}$$

$$\left(a_k\,e^{-i\,\omega_k\,t}\right)^\dagger=a_k^\dagger\,e^{i\,\omega_k\,t}$$

Llegando al mismo resultado.

$$e^{it\,H}\,\hat{\phi}_{(\vec{x})}\,e^{-it\,H}=e^{it\,H}\left(\int\frac{dk}{2\pi}\frac{1}{\sqrt{2\omega_k}}\left(a_k\,e^{i\,\vec{k}\cdot\vec{x}}+a_k^\dagger\,e^{-i\,\vec{k}\cdot\vec{x}}\right)\right)e^{-it\,H}$$

$$e^{it\,H}\,\hat{\phi}_{(\vec{x})}\,e^{-it\,H}=\int\frac{dk}{2\pi}\frac{1}{\sqrt{2\omega_k}}e^{it\,H}\left(a_k\,e^{i\,\vec{k}\cdot\vec{x}}+a_k^\dagger\,e^{-i\,\vec{k}\cdot\vec{x}}\right)\,e^{-it\,H}$$

$$e^{it\,H}\,\hat{\phi}_{(\vec{x})}\,e^{-it\,H}=\int\frac{dk}{2\pi}\frac{1}{\sqrt{2\omega_k}}\left(\left(e^{it\,H}a_k\,e^{-it\,H}\right)e^{i\,\vec{k}\cdot\vec{x}}+\left(e^{it\,H}a_k^\dagger\,e^{-it\,H}\right)e^{-i\,\vec{k}\cdot\vec{x}}\right)$$

$$e^{it\,H}\,\hat{\phi}_{(\vec{x})}\,e^{-it\,H}=\int\frac{dk}{2\pi}\frac{1}{\sqrt{2\omega_k}}\left(\left(a_k\,e^{-i\,\omega_k\,t}\right)e^{i\,\vec{k}\cdot\vec{x}}+\left(a_k^\dagger\,e^{i\,\omega_k\,t}\right)e^{-i\,\vec{k}\cdot\vec{x}}\right)$$

$$\boxed{\boldsymbol{e^{it\,H}\,\hat{\phi}_{(\vec{x})}\,e^{-it\,H}=\int\frac{dk}{2\pi}\frac{1}{\sqrt{2\omega_k}}\left(a_k\,e^{-i\,\omega_k\,t+i\,\vec{k}\cdot\vec{x}}+a_k^\dagger\,e^{\,i\,\omega_k\,t-i\,\vec{k}\cdot\vec{x}}\right)}}$$

<u>EJERCICIOS 2 (59:05)</u>

El operado de evolución en la imagen de interacción es:

$$\boldsymbol{U_{I\,(t)} = e^{i\,t\,H_0} e^{-i\,t\,H}}$$

2.1 Comprobar que

$$\boldsymbol{U_{I\,(t,t')} = e^{i\,t\,H_0}\; e^{-i\,(t-t')\,H} e^{-i\,t'\,H_0}}$$

$$U_{I\,(t,t')} = U_{I\,(t)} U^{\dagger}_{I\,(t')}$$

$$U_{I\,(t,t')} = \left(e^{i\,t\,H_0}\; e^{-i\,t\,H}\right)\left(e^{i\,t'\,H_0}\; e^{-i\,t'\,H}\right)^{\dagger}$$

$$U_{I\,(t,t')} = \left(e^{i\,t\,H_0}\; e^{-i\,t\,H}\right)\left(e^{i\,t'\,H}\; e^{-i\,t'\,H_0}\right)$$

$$\boxed{\boldsymbol{U_{I\,(t,t')} = e^{i\,t\,H_0} e^{-i\,(t-t')\,H} e^{-i\,t'\,H_0}}}$$

2.2 Comprobar que $U_{I\,(t,t')}$ es solución de

$$\boldsymbol{i\,\partial_t U_{I\,(t,t')} = H'_{I\,(t)} U_{I\,(t,t')}}$$

$$U_{I\,(t,t')} = e^{i\,t\,H_0} e^{-i\,(t-t')\,H} e^{-i\,t'\,H_0} = e^{-i\,t\,(H-H_0)} e^{i\,t'\,(H-H_0)}$$

$$\partial_t U_{I\,(t,t')} = -i(H-H_0) e^{-i\,t\,(H-H_0)} e^{i\,t'\,(H-H_0)} = -i(H-H_0) U_{I\,(t,t')}$$

$$i\,\partial_t U_{I\,(t,t')} = (H-H_0) U_{I\,(t,t')}$$

Recordando que $H' = H - H_0$

$$\boxed{\boldsymbol{i\,\partial_t U_{I\,(t,t')} = H'\, U_{I\,(t,t')}}}$$

2.3 Verificar que cumple (estos operadores conforman un grupo) para $t_1 > t_2 > t_3$

$$\boldsymbol{U_{I\,(t_1,t_2)} U_{I\,(t_2,t_3)} = U_{I\,(t_1,t_3)}}$$

$$U_{I\,(t_1,t_2)} = e^{i\,t_1\,H_0} e^{-i\,(t_1-t_2)\,H} e^{-i\,t_2\,H_0}$$

$$U_{I\,(t_2,t_3)} = e^{i\,t_2\,H_0} e^{-i\,(t_2-t_3)\,H} e^{-i\,t_3\,H_0}$$

$$U_{I\,(t_1,t_2)} U_{I\,(t_2,t_3)} = e^{i\,t_1\,H_0} e^{-i\,(t_1-t_2)\,H}\; e^{-i\,t_2\,H_0} e^{i\,t_2\,H_0}\; e^{-i\,(t_2-t_3)\,H} e^{-i\,t_3\,H_0}$$

$$U_{I\,(t_1,t_2)} U_{I\,(t_2,t_3)} = e^{i\,t_1\,H_0} e^{-i\,(t_1-t_2)\,H}\; 1\; e^{-i\,(t_2-t_3)\,H} e^{-i\,t_3\,H_0}$$

$$U_{I\,(t_1,t_2)} U_{I\,(t_2,t_3)} = e^{i\,t_1\,H_0} e^{-i\,(t_1-t_2)\,H - i\,(t_2-t_3)\,H}\; e^{-i\,t_3\,H_0}$$

$$U_{I\,(t_1,t_2)} U_{I\,(t_2,t_3)} = e^{i\,t_1\,H_0} e^{-i\,(t_1-t_3)\,H}\; e^{-i\,t_3\,H_0}$$

2.4 Demostrar que

$$\boldsymbol{U_{I\,(t_1,t_3)} U^{\dagger}_{I\,(t_2,t_3)} = U_{I\,(t_1,t_2)}}$$

$$\left(U_{I\,(t_2,t_3)}\right)^{\dagger} = \left(e^{i\,t_2\,H_0} e^{-i\,(t_2-t_3)\,H} e^{-i\,t_3\,H_0}\right)^{\dagger} = e^{i\,t_3\,H_0} e^{i\,(t_2-t_3)\,H} e^{-i\,t_2\,H_0}$$

$$U_{I\,(t_1,t_3)}\left(U_{I\,(t_2,t_3)}\right)^{\dagger} = \left(e^{i\,t_1\,H_0} e^{-i\,(t_1-t_3)\,H}\; e^{-i\,t_3\,H_0}\right)\left(e^{i\,t_3\,H_0} e^{i\,(t_2-t_3)\,H} e^{-i\,t_2\,H_0}\right)$$

$$U_{I\,(t_1,t_3)}\left(U_{I\,(t_2,t_3)}\right)^{\dagger} = e^{i\,t_1\,H_0} e^{-i\,(t_1-t_3)\,H}\; 1\; e^{i\,(t_2-t_3)\,H} e^{-i\,t_2\,H_0}$$

$$U_{I\,(t_1,t_3)}\left(U_{I\,(t_2,t_3)}\right)^{\dagger} = e^{i\,t_1\,H_0} e^{-i\,(t_1-t_3)\,H+i\,(t_2-t_3)\,H}\; e^{-i\,t_2\,H_0}$$

$$U_{I\,(t_1,t_3)}\left(U_{I\,(t_2,t_3)}\right)^{\dagger} = e^{i\,t_1\,H_0} e^{-i\,(t_1-t_2)\,H}\; e^{-i\,t_2\,H_0}$$

$$\boxed{\boldsymbol{U_{I\,(t_1,t_3)} U^{\dagger}_{I\,(t_2,t_3)} = U_{I\,(t_1,t_2)}}}$$